

the line $\lambda=3995$ is $12+24-8.5=27.5$ volts, while the corresponding quantity for the C^+ line $\lambda=4267$ is $7+24-6.1=24.9$ volts. Thus the nitrogen lines present not only no difficulty to the theory of steric factor, but they probably justify the theory. We must, however, wait till the I.P. of C, N, and N^+ are accurately determined.

*Physics Department,
Allahabad University:
1925 August 14.*

Note on Rosseland's Integral for the Stellar Absorption Coefficient.
By Professor E. A. Milne.

In a recent paper * Rosseland has shown that if $k(\nu)$ is the mass absorption coefficient of a gas in frequency ν , the correct mean absorption coefficient \bar{k} to use in calculating the net flux of radiation is given by the formula

$$\frac{1}{\bar{k}} = \int_0^\infty \frac{1}{k(\nu)} \phi_\nu d\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\phi_\nu = \frac{\partial B_\nu}{\partial T} \bigg/ \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu$$

and

$$B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1},$$

in the usual notation. For brevity we shall call the mean value \bar{k} derived from $k(\nu)$ in this way the "Rosseland mean-value" of $k(\nu)$.

When a theoretical calculation of \bar{k} is being undertaken, from atomic theory, it is usual to assume that the gas is made up of one particular element, *e.g.* iron, and to determine \bar{k} for pure iron vapour. It is then natural to suppose that a mixture of elements will have an absorption coefficient which is a mean amongst the calculated absorption coefficients of the separate constituents.

This, however, is not necessarily true. It is partly the object of this note to point out that when the Rosseland mean absorption coefficient is calculated for each constituent separately, the Rosseland mean absorption coefficient for the mixture in general exceeds that formed by combining the separate Rosseland means according to the proportions in which the constituents are present, and, moreover, will not even necessarily lie between the greatest and least of these.

Let a_1, \dots, a_n be the proportions by mass of the separate constituents, $k_1(\nu), \dots, k_n(\nu)$ their absorption coefficients. Then each constituent has a separate Rosseland mean of the type

$$\frac{1}{\bar{k}_r} = \int \frac{1}{k_r(\nu)} \phi_\nu d\nu,$$

* *M.N.*, **84**, 525, 1924.

whilst the mixture has an absorption coefficient $k(\nu)$ for frequency ν given by

$$k(\nu) = a_1 k_1(\nu) + \dots + a_n k_n(\nu).$$

The first result stated above follows immediately from the inequality (see Appendix),

$$\frac{1}{\int_0^\infty \frac{\phi_\nu d\nu}{a_1 k_1(\nu) + \dots + a_n k_n(\nu)}} \geq \frac{a_1}{\int_0^\infty \frac{\phi_\nu d\nu}{k_1(\nu)}} + \dots + \frac{a_n}{\int_0^\infty \frac{\phi_\nu d\nu}{k_n(\nu)}} \quad (2)$$

which shows that

$$k \geq a_1 k_1 + \dots + a_n k_n.$$

The inequality becomes an equality only when $k_1(\nu), \dots, k_n(\nu)$ are all proportional to one another for all ν , i.e. are the same function except for a multiplying constant.

The second of the results enumerated above can be seen from an example. We take an artificial case. Determine the frequency ν_1 for which

$$\int_0^{\nu_1} \phi_\nu d\nu = \int_{\nu_1}^\infty \phi_\nu d\nu = \frac{1}{2}.$$

Suppose there are two constituents. Let the absorption coefficients be given by the relations

$$\begin{aligned} k_1(\nu) &= \epsilon, & k_2(\nu) &= A, & (0 < \nu < \nu_1) \\ k_1(\nu) &= A, & k_2(\nu) &= \epsilon, & (\nu_1 < \nu < \infty), \end{aligned}$$

where ϵ and A are constants. It is clear that a mixture of equal masses of the two will have an absorption coefficient $k(\nu)$ given by

$$k(\nu) = \frac{1}{2}(\epsilon + A),$$

constant throughout the spectrum. The Rosseland mean of this is, of course, $k = \frac{1}{2}(\epsilon + A)$ also. But the Rosseland mean for either constituent by itself is given by

$$\frac{1}{k_1} = \frac{1}{k_2} = \frac{1}{2} \left(\frac{1}{\epsilon} + \frac{1}{A} \right),$$

or

$$k_1 = k_2 = 2\epsilon A / (\epsilon + A).$$

Since

$$\frac{1}{2}(\epsilon + A) > \frac{2\epsilon A}{\epsilon + A},$$

we have not only

$$k > \frac{1}{2}(k_1 + k_2),$$

in accordance with (2), but also

$$\begin{aligned} k &> k_1, \\ k &> k_2. \end{aligned}$$

The Rosseland mean absorption coefficient of this mixture not only exceeds the mean of the Rosseland means of the separate constituents,

but exceeds either separately. If ϵ is small compared with A , we have $k_1 = k_2 = 2\epsilon$, approximately, whilst $k = \frac{1}{2}A$. The ratio is $\frac{1}{2}A/\epsilon$, which may be as large as we please.

We see that when Rosseland means for pure elements are used for a mixture, there is danger of getting too small a coefficient. The Rosseland mean emphasises the smallest values of the absorption coefficient whose mean is being taken. It is the smallest values that have a predominant influence on the mean. But in a mixture, if the small values for one constituent occur in the same spectral region as the large values for another, the small values lose all their importance, and the absorption coefficients calculated for the separate constituents lose most of their significance as regards the absorption coefficient of the mixture.

Suppose, on the other hand, we consider what may be called the arithmetic mean absorption coefficient, k' , defined by

$$k' = \int_0^\infty k(\nu) \phi_\nu d\nu.$$

We have in all cases

$$k' = a_1 k_1' + \dots + a_n k_n'.$$

In consequence of the inequality

$$\int_0^\infty k(\nu) \phi_\nu d\nu \geq \int_0^\infty \frac{1}{k(\nu)} \phi_\nu d\nu \quad (3)$$

which holds in virtue of Schwarz's Inequality, since

$$\int_0^\infty \phi_\nu d\nu = 1,$$

we see that

$$k' \geq k.$$

Rosseland and Eddington have drawn attention to this result, namely, that the arithmetic mean gives in general too large values. The inequality becomes an equality only if $k(\nu)$ is a constant independent of ν .

For a mixture, in the proportions a_1, \dots, a_n , our inequalities give altogether

$$\begin{aligned} & \int_0^\infty [a_1 k_1(\nu) + \dots + a_n k_n(\nu)] \phi_\nu d\nu \\ & \geq \frac{1}{\int_0^\infty \frac{\phi_\nu d\nu}{a_1 k_1(\nu) + \dots + a_n k_n(\nu)}} \geq \frac{a_1}{\int_0^\infty \frac{\phi_\nu d\nu}{k_1(\nu)}} + \dots + \frac{a_n}{\int_0^\infty \frac{\phi_\nu d\nu}{k_n(\nu)}} k \quad (4) \end{aligned}$$

i.e.

$$a_1 k_1' + \dots + a_n k_n' \geq k \geq a_1 k_1 + \dots + a_n k_n.$$

We now observe that if it happens that the absorption coefficient of the whole mixture,

$$k(\nu) = a_1 k_1(\nu) + \dots + a_n k_n(\nu),$$

considered as a function of ν , is a constant independent of ν , whilst the same is not true of $k_1(\nu), \dots k_n(\nu)$ separately, then not only will the arithmetic mean be equal to the Rosseland mean but (1) the arithmetic means of the separate constituents combined in the proportions $a_1, \dots a_n$ will give the correct value of the coefficient for the mixture, namely, the common value of the arithmetic mean and the Rosseland mean for the mixture; (2) the same will not be true of the Rosseland means of the separate constituents, which, when combined in the proportions $a_1, \dots a_n$, will give too small a value; (3) the arithmetic means of the separate constituents will depend on the precise weighting factor ϕ_ν , but the combination of the arithmetic means will be independent of ϕ_ν .

Thus, if it happens that the separate opacities combine to give approximately "grey" absorption, there is a considerable advantage in calculating with the arithmetic means instead of the Rosseland means of the separate constituents. For in these circumstances we get the right result by combining the arithmetic means of the separate constituents as they stand, and the result is, moreover, independent of the weighting factor ϕ_ν used in the process.

The importance of this is in its application to the photospheric layers of the sun, and indeed of all stars which show an approximately black-body continuous spectrum. If the spectrum is black body, the absorption coefficient must be the same for all frequencies,* and we have the circumstances just contemplated. The measurements of Fabry and Buisson† and of H. H. Plaskett‡ show that the sun's continuous spectrum is black body in the region 3000–6700 Å. Moreover, an approximately constant absorption coefficient is indicated by the observed intensity-distribution over the sun's disc in the separate wavelengths. In a recent paper on absorption coefficients in the photospheric layers I showed that the opacity principally arises from photo-electric absorption.§ Since observation shows the sun's photospheric material to absorb as a "grey" body, it can only be the case that this arises from the superposition of regions of continuous photo-electric absorption, each sharply bounded on the long wave-length side and stretching indefinitely towards the higher frequencies. In each separate absorption region thus combined the absorption coefficient $k_\nu(\nu)$ varies roughly as ν^{-3} , and so decreases as ν increases; but this will, at least in part, be compensated by the fact that the higher frequencies can be absorbed by a greater number of atomic levels, the lower ones only by the atoms with the series electron in outer orbits. The smoothing-out of these discontinuously bounded tracts of absorption probably occurs in virtue of superpositions of such tracts from a number of different atoms.

In my paper, for reasons of convenience, I used the arithmetic mean

* If the coefficient was not constant, the existence of a temperature gradient in the photospheric layers would give rise to departures from a black-body spectrum: where the coefficient became smaller, we should see deeper and the emergent radiation would be more intense, and conversely.

† *Comptes Rendus*, **175**, 156, 1922.

‡ *Pub. Domin. Astrophys. Obs.*, **2**, 213, 1923.

§ *M.N.*, **85**, 768, 1925.

instead of the theoretically more correct Rosseland mean. But we see now that whatever the physical origin of the observed constancy of $k(\nu)$ for different ν for the sun's layers, it justifies the use of an arithmetic mean as equivalent to the Rosseland mean in determining the value of the absorption coefficient. Of course, one does not attempt to introduce actual proportions $a_1, \dots a_n$; but the observed constancy gives some confidence that the error committed in using the arithmetic mean is relatively slight, so that the coefficients thus calculated do afford a true indication of the opacity of the mixture.

Whether a similar state of affairs holds in the interior of a star is more difficult to ascertain. Strictly speaking, the Rosseland mean absorption coefficient cannot be evaluated until we know the proportions in which the different constituents are present, unless the different constituents have absorption coefficients which all behave in the same way considered as functions of ν . Unless the latter holds, it is difficult to obtain any algebraically simple formula for the mean absorption coefficient, since the integrand involves the sum of the opacities in the denominator.

In his recent valuable paper * on the stellar absorption coefficient, Rosseland has effected the integration by increasing certain exponential factors until they are unity. This in effect reduces all the separate absorption coefficients to a single function of ν , thus making the integration possible, but as the result is then only an upper limit he has sacrificed some of the accuracy introduced by using his own mean value instead of an arithmetic mean. For iron, a pure substance, Rosseland obtained $k=10$ for Eddington's data for Capella. In my own paper † I obtained $k=6.85$, or $k=8.8$ on correcting for the empirical value of Kramer's " g' ." I gave reasons for considering that the methods of summation and integration automatically accounted for part of Kramer's factor $g'=2.3$, requiring only $g'=1.3$. Had Rosseland used $g=1.3$ he would have obtained $k=5.7$. My value, $k=8.8$, is larger than this. Rosseland's is, however, the theoretical upper limit for the pure element. There were other minor points of difference between our calculations besides our different ways of averaging.

The question whether Rosseland's calculations or mine more accurately represent the theoretical absorption coefficient for the physical conditions in a stellar interior turns largely on whether the general run of the total opacity, in the important region, is given by a formula of the type

$$k(\nu) \propto \nu^{-3} e^{h\nu/kT},$$

as used by Rosseland, or whether the different photo-electric limits of the different constituents in their different excited states so combine as to give an absorption coefficient approximately independent of ν . However improbable it may appear *a priori*, this seems to be so for the photospheric layers—say, at a temperature of 10^4 degrees. The question is whether the same holds at 10^7 degrees.

* *Astrophys. Journ.*, **61**, 424, 1925.

† *M.N.*, **85**, 750, 1925.

Appendix.

The first part of inequality (4) follows immediately from the well-known Schwarz's Inequality for integrals. The second part I have not seen explicitly stated. To prove it, it is clearly sufficient to prove that

$$\frac{\int \frac{dx}{f(x)+g(x)}}{\int \frac{dx}{f(x)}} \geq \frac{\int \frac{dx}{f(x)}}{\int \frac{dx}{f(x)}} + \frac{\int \frac{dx}{g(x)}}{\int \frac{dx}{g(x)}},$$

the integrals being taken through some given range. Put

$$1/f(x) = F(x), \quad 1/g(x) = G(x).$$

Then we have to prove that

$$\frac{\int F(x)G(x)dx}{\int F(x)+G(x)} \geq \frac{\int F(x)dx}{\int F(x)} + \frac{\int G(x)dx}{\int G(x)},$$

i.e. if we suppose that $\int F(x)dx$ and $\int G(x)dx$ are both positive, that

$$\left(\int F(x)dx\right)\left(\int G(x)dx\right) \geq \left(\int \frac{F(x)G(x)}{F(x)+G(x)}dx\right)\left(\int [F(x)+G(x)]dx\right) \quad (5)$$

It is clearly sufficient to prove that

$$(\sum F_r \Delta_r)(\sum G_r \Delta_r) \geq \left(\sum \frac{F_r G_r}{F_r + G_r} \Delta_r\right)(\sum (F_r + G_r) \Delta_r),$$

i.e. taking the coefficient of $\Delta_r \Delta_s$, that

$$F_r G_s + F_s G_r \geq \frac{F_r G_r (F_s + G_s)}{F_r + G_r} + \frac{F_s G_s (F_r + G_r)}{F_s + G_s}.$$

This reduces to

$$(F_r G_s - F_s G_r)^2 \geq 0,$$

provided $F+G$ is always positive. The inequality therefore holds unless $F/G = \text{constant}$, in which case it reduces to an equality.

Applying Schwarz's Inequality once again, we have from (5)

$$\left(\int F(x)dx\right)\left(\int G(x)dx\right) \geq \left(\int \frac{F(x)G(x)}{F(x)+G(x)}dx\right)\left(\int [F(x)+G(x)]dx\right) \geq \left(\int \sqrt{F(x)G(x)}dx\right)^2 \quad (6)$$

The inequality we have established is thus more delicate than Schwarz's, which consists of the first and third terms in (6).