

Gravitational lensing with $f(\chi) = \chi^{3/2}$ gravity in accordance with astrophysical observations

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ABSTRACT

In this article, we perform a second order perturbation analysis of the gravitational metric theory of gravity $f(\chi) = \chi^{3/2}$ developed by Bernal et al. We show that the theory accounts in detail for two observational facts: (1) the phenomenology of flattened rotation curves associated with the Tully–Fisher relation observed in spiral galaxies, and (2) the details of observations of gravitational lensing in galaxies and groups of galaxies, without the need of any dark matter. We show how all dynamical observations on flat rotation curves and gravitational lensing can be synthesized in terms of the empirically required metric coefficients of any metric theory of gravity. We construct the corresponding metric components for the theory presented at second order in perturbation, which are shown to be perfectly compatible with the empirically derived ones. It is also shown that under the theory being presented, in order to obtain a complete full agreement with the observational results, a specific signature of Riemann’s tensor has to be chosen. This signature corresponds to the one most widely used nowadays in relativity theory. Also, a computational program, the Metric EXtended-gravity Incorporated through a Computer Algebraic System (MEXICAS) code, developed for its usage in the Computer Algebraic System Maxima for working out perturbations on a metric theory of gravity, is presented and made publicly available.

Key words: gravitation – gravitational lensing: strong – gravitational lensing: weak.

1 INTRODUCTION

When Einstein introduced his theory of general relativity, an astrophysical prediction for the motion of the planet Mercury (a massive particle) through its orbit was made (Einstein 1916). The second step was to test general relativity through the deflection of light (massless particles) coming from stars appearing near the Sun’s limb during a solar eclipse (Dyson, Eddington & Davidson 1920). Both observations constituted the first coherent steps towards the solid foundation of general relativity, a theory capable of describing gravitation through a correct relativistic description.

In this sense, any metric theory of gravity must be compatible with both kinds of observations, the dynamical ones for massive particles and the observations of the deflection of light for massless particles. The correct approach is extensively described in the monograph by Will (1993) where it is shown that when working with the weak field limit of a relativistic theory of gravity in a static spherically space–time, the dynamics of massive particles determine the functional form of the time component of the metric, while the deflection of light determines the form of the radial one (see also Will 2006, and references therein).

To order of magnitude and through a first perturbation analysis, Bernal et al. (2011b) have shown that it is possible to recover flat rotation curves and the Tully–Fisher relation (i.e. a MONDian-like weak field limit) from a metric theory of gravity, which includes the mass of the system in the gravitational field’s action. Such limit is of high astrophysical relevance at the scales of galaxies, where Modified Newtonian Dynamics (MOND) accurately describes the rotation curves of spiral galaxies and the Tully–Fisher relation without the need of dark matter (see e.g. Milgrom 1983; Famaey & McGaugh 2012). In this article, we show the strength of the calculations made by Bernal et al. (2011b) by doing an extensive analysis from perturbation theory for a static spherically symmetric metric and show that in the weak field limit our results are in perfect agreement not only with the Tully–Fisher relation, but are also in exact accordance with observations of gravitational lensing over a wide range of astrophysical scales.

Extensions to Einstein’s general theory of relativity have been proposed since the publication of the theory itself (see e.g. Schimming & Schmidt 1990). However, it has not been until recent times that observations at different mass and length scales have concluded that in order to keep Einstein’s field equations valid, unknown dark matter and energy entities need to be added to the theory. In this article, a complementary approach is taken where the existence of these unknown dark entities is not required. We show

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the theory built by Bernal et al. (2011b) to be in accordance not only with the very well established observations of the dynamics of massive particles through the Tully–Fisher relation, but also with the dynamics of massless particles through the bending of light as astrophysically observed.

Mendoza & Rosas-Guevara (2007) and Rosas-Guevara (2006) showed for the first time that metric theories of gravity are capable of producing more deflection of light than the one produced by Einstein’s general relativity. This was done using the metric theory of gravity constructed by Sobouti (2007). The implications of this result invalidated the so-called no-go theorem for metric $f(R)$ theories of gravity proposed by Soussa & Woodard (2003); Soussa (2003). Furthermore, in this work, we show that it is possible to explain the observed gravitational lensing for galaxies, and groups of galaxies without the need of invoking dark matter. Developments, by see e.g. Capozziello et al. (2006), Horváth et al. (2012), Nzioki et al. (2011), on weak and strong lensing regimes of extended metric theories of gravity have followed the work by Mendoza & Rosas-Guevara (2007) but are not of general validity with respect to different astrophysical observations.

Testing any metric theory of gravity against observations can be cumbersome. From an action principle one must derive field equations, which in principle, have to be solved for e.g. in spherically symmetric space–times. The solutions to this lead to metric coefficients which in turn, with the use of the geodesic equation, yield orbits for massive and massless particles, to be then compared against astrophysical observations. These last are varied and diverse e.g., centrifugal equilibrium orbits at a variety of radii, for systems having total masses spanning several orders of magnitude, and the observed shears and caustic positions of gravitational lensing observations.

Fortunately, we have derived a much more direct and generic approach. First, dynamical observations regarding the amplitudes of galactic flat rotation curves satisfy a well-known scaling with the fourth root of the total baryonic content: the Tully–Fisher relation. To second order in perturbations of the velocity measured in units of the speed of light, this can be shown to imply a definite empirical prescription for the time component of any metric theory not requiring dark matter. Secondly, we show that all gravitational lensing observations on elliptical and spiral galaxies, as well as for groups of galaxies can be synthesized as the requirement for the same isothermal total matter distribution as needed to explain the observed spiral rotation curves and dynamics about elliptical galaxies, if one assumes Einstein’s general relativity. From studying directly the lens equation in general relativity, this implies a bending angle which is independent of the impact parameter, and which scales with the square root of the total baryonic mass of a system. It can then be shown that this, in combination with the empirical time component of the metric mentioned above, leads to a fixed empirical prescription for the radial component, for any metric theory not requiring dark matter.

Thus, we synthesize all dynamical and gravitational lensing astrophysical observations at galactic and galaxy group scales, into empirical time and radial metric components of a spherically symmetric metric given at second order in perturbation. It is through comparing the above to perturbed metric coefficients to the same order coming from the metric theory treated in this paper that we are able to show its full compatibility with all relevant dynamical and gravitational lensing astrophysical observations.

The article is organized as follows. In Section 2, the concept of weak field limit for a static spherically symmetric space–time is established and we define the relevant orders of perturbation to be

used throughout the article. In Section 3, we perturb the vacuum field equations of the metric theory built by Bernal et al. (2011b) and show that for a point mass source they closely resemble the ones usually adopted in $f(R)$ gravity in vacuum. However, these equations slightly differ under the approximations of the mass and length scales associated with galaxies and groups of galaxies – where gravity is expected to differ from Einstein’s general relativity in the absence of any dark matter component. In Section 4, we obtain the solution for the Ricci scalar up to the second order from the perturbed field equations and discuss the importance of the signature in the Riemann tensor to yield the correct results. In Section 5, we obtain the coefficients of the metric up to the second order in perturbation. In Section 6, we obtain the metric coefficients up to the second order in an empirical way, without reference to any specific metric theory of gravity, using the dynamical phenomenology of galaxies and groups of galaxies and the gravitational lensing produced by these objects. In that section, we also compare the metric coefficients obtained in 5 with those empirically obtained and show full consistency. Finally in Section 7, we discuss our results.

2 THE WEAK FIELD LIMIT

An excellent account of perturbation theory applied to metric theories of gravity (in particular general relativity) can be found in the monograph written by Will (1993). More recently, Capozziello & Stabile (2009) have developed a perturbation analysis technique useful when dealing with lenses in $f(R)$ gravity. In this Section, we define the relevant properties of the perturbation theory having in mind applications to the metric theory developed by Bernal et al. (2011b).

Let us consider a fixed point mass M at the centre of coordinates generating a gravitational field. Under these considerations, the space–time is static and its spherically symmetric metric $g_{\mu\nu}$ is generated by the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2 + g_{11} dr^2 - r^2 d\Omega^2. \quad (1)$$

In the previous equation and in what follows, Einstein’s summation convention over repeated indices is used. Greek indices take values 0, 1, 2, 3 and Latin ones 1, 2, 3. As such, in spherical coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$, where c is the speed of light, t is the time coordinate, r the radial one, and θ and φ are the polar and azimuthal angles, respectively. Also, the angular displacement $d\Omega^2 := d\theta^2 + \sin^2\theta d\varphi^2$. Due to the symmetry of the problem, the unknown functions g_{00} and g_{11} are functions of the radial coordinate r only. Note also that We choose a $(+, -, -, -)$ signature for the space–time metric, which we maintain throughout the article.

The radial component of the geodesic equations

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\beta\lambda}^\alpha \frac{dx^\beta}{ds} \frac{dx^\lambda}{ds} = 0, \quad (2)$$

for the metric (1) in the weak field limit, i.e. when the speed of light $c \rightarrow \infty$, is given by

$$\frac{1}{c^2} \frac{d^2 r}{dt^2} = \frac{1}{2} g^{11} g_{00,r}, \quad (3)$$

where the subscript $(\cdot)_{,r} := \partial/\partial r$ denotes the derivative with respect to the radial coordinate. In the above relation, we have assumed that for the weak field limit $ds = c dt$ and since the velocity $v \ll c$ then $v^i \ll dx^0/dt$ with $v^i := (dr/dt, r d\theta/dt, r \sin\theta d\varphi/dt)$. In the strong $c \rightarrow \infty$ limit, both sides of the above equation vanish simultaneously. Thus, the condition of the right-hand side of equation (3) vanishing in the $v \ll c$ provides a consistency check on the results

of the following sections, where a perturbative solution to the metric is developed.

In this limit, a particle bound to a circular orbit about the mass M experiences a centrifugal radial acceleration given by

$$\frac{d^2 r}{dt^2} = -\frac{v^2}{r}, \quad (4)$$

for a circular or tangential velocity v . The preceding equation is a kinematical relation of general validity and does not introduce any particular assumption of the gravitational theory.

When material particles are used as test particles in the weakest limit of the theory, the metric takes the form (see e.g. Landau & Lifshitz 1975):

$$g_{00} = 1 + \frac{2\phi}{c^2}, \quad g_{11} = -1, \\ g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad (5)$$

for a Newtonian gravitational potential ϕ . The above equations are only used when analysing the motion of material particles when gravity is very weak (see e.g. Will 1993). In order to demand greater accuracies of the theory and to recover exact results for the motion of massless particles, i.e. to accurately describe the bending of light rays, the following term in the expansion of g_{11} must also be considered.

For a particle on circular motion about the mass M in the weak field limit, the lowest order of the theory is obtained when the left-hand side of equation (3) is of the order of v^2/c^2 and when the right-hand side is of the order of ϕ/c^2 . Both are just orders $\mathcal{O}(1/c^2)$ of the theory, or simply $\mathcal{O}(2)$. As such, when lower or higher order corrections of the theory are introduced we will use the notation $\mathcal{O}(n)$ for $n = 0, 1, 2, \dots$ meaning $\mathcal{O}(0)$, $\mathcal{O}(c^{-1})$, $\mathcal{O}(c^{-2})$, \dots , respectively.

Having in mind further astrophysical applications (of motion of material particles and bending of light – massless particles), we expand the metric $g_{\mu\nu}$ about a flat Minkowski metric $\eta_{\mu\nu} := \text{diag}(1, -1, -1, -1)$ up to the second order in time and radial position in such a way that

$$g_{00} = g_{00}^{(0)} + g_{00}^{(2)} = 1 + g_{00}^{(2)} + \mathcal{O}(4), \\ g_{11} = g_{11}^{(0)} + g_{11}^{(2)} = -1 + g_{11}^{(2)} + \mathcal{O}(4), \\ g_{22} = g_{22}^{(0)} = -r^2, \\ g_{33} = g_{22} \sin^2 \theta, \quad (6)$$

where the superscript (p) denotes the order $\mathcal{O}(p)$ at which a particular quantity is approximated. From equations (6) it follows that the contravariant metric components are given by

$$g^{00} = g^{00(0)} + g^{00(2)} = 1 - g_{00}^{(2)} + \mathcal{O}(4), \\ g^{11} = g^{11(0)} + g^{11(2)} = -1 - g_{11}^{(2)} + \mathcal{O}(4), \\ g^{22} = g^{22(0)} = -1/r^2, \\ g^{33} = g^{22} / \sin^2 \theta. \quad (7)$$

In fact, to the lowest order of perturbation, we need to find the time $g_{00}^{(2)}$ and radial $g_{11}^{(2)}$ metric components up to the second order to compare with the astrophysical observations of material particles and bending of light (Will 1993, 2006). Note that in keeping with the assumption of spherical symmetry for the matter configurations to be studied, we consider no perturbations on the angular terms of the metric. This assumption thus limits the applicability of all our following results to systems not far from spherical symmetry, e.g.

the spheroidal elliptical galaxies about which gravitational lenses are often detected.

3 FIELD EQUATIONS

For the case of a point-mass source generating a gravitational field, Bernal et al. (2011b) have proposed an extended gravitational field's action in the metric approach given by

$$S_f = -\frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g} d^4 x, \quad (8)$$

for any arbitrary dimensionless function $f(\chi)$ of the dimensionless Ricci scalar:

$$\chi := L_M^2 R, \quad (9)$$

where R is the standard Ricci scalar and

$$L_M = \zeta r_g^{1/2} l_M^{1/2}, \quad (10)$$

is a length scale with

$$r_g := \frac{GM}{c^2}, \quad l_M := \left(\frac{GM}{a_0} \right)^{1/2}, \quad (11)$$

with l_M the mass-length scale of the system defined by Mendoza et al. (2011), $a_0 := 1.2 \times 10^{-10} \text{ m s}^{-2}$ is Milgrom's acceleration constant (see e.g. Famaey & McGaugh 2012, and references therein) and ζ is a coupling constant of order one which has to be calibrated through astrophysical observations. This $f(\chi)$ theory was constructed through the inclusion of a_0 as a fundamental physical constant, which has been shown to be of astrophysical and cosmological relevance (see e.g. Hernandez et al. 2010; Bernal et al. 2011a,b; Mendoza et al. 2011; Hernandez & Jiménez 2012; Hernandez, Jiménez & Allen 2012; Mendoza 2012; Carranza, Mendoza & Torres 2013). Equation (8) is understood as a particular case of a fuller formulation where the details of the mass distribution appear inside of the action integral, in such a way that for a fixed point mass, the result is the action (8), as will be more fully discussed towards the end of this section.

Following the description of Bernal et al. (2011b), the matter action takes its ordinary form:

$$S_m = -\frac{1}{2c} \int \mathcal{L}_m \sqrt{-g} d^4 x, \quad (12)$$

with \mathcal{L}_m the Lagrangian matter density of the system. The null variation of the complete action, i.e. $\delta(S_f + S_m) = 0$, with respect to the metric $g_{\mu\nu}$ yields the following field equations:

$$f'(\chi) \chi_{\mu\nu} - \frac{1}{2} f(\chi) g_{\mu\nu} - L_M^2 (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta) f'(\chi) \\ = \frac{8\pi G L_M^2}{c^4} T_{\mu\nu}, \quad (13)$$

where the Laplace–Beltrami operator has been written as $\Delta := \nabla^\mu \nabla_\mu$, the prime denotes the derivative with respect to the argument and the energy–momentum tensor $T_{\mu\nu}$ is defined through the standard relation $\delta S_m = -(1/2c) T_{\alpha\beta} \delta g^{\alpha\beta}$. Also, in equation (13), the dimensionless Ricci tensor is defined as

$$\chi_{\mu\nu} := L_M^2 R_{\mu\nu}, \quad (14)$$

where $R_{\mu\nu}$ is the standard Ricci tensor.

The trace of equations (13) is

$$f'(\chi) \chi - 2f(\chi) + 3L_M^2 \Delta f'(\chi) = \frac{8\pi G L_M^2}{c^4} T, \quad (15)$$

where $T := T^\alpha_\alpha$.

To order of magnitude approximation, where $d/d\chi \approx 1/\chi$, $\Delta \approx -1/r^2$ and the mass density $\rho \approx M/r^3$, Bernal et al. (2011b) have shown that the trace (15) equation takes the following form:

$$\chi^b(b-2) - 3bL_M^2 \frac{\chi^{(b-1)}}{r^2} \approx \frac{8\pi GML_M^2}{c^2 r^3}. \quad (16)$$

for a power-law form:

$$f(\chi) = \chi^b. \quad (17)$$

As shown by Bernal et al. (2011b), the third term on the left-hand side of equation (15) dominates over the first two when the radius of curvature $R_c \approx R^{-1/2}$ of space-time is such that $R_c \gg r$ and so, this corresponds to the region where MONDian effects are expected to appear.

Bernal et al. (2011b) and Mendoza (2012) have shown that the function $f(\chi)$ must satisfy the following limits:

$$f(\chi) = \begin{cases} \chi, & \text{when } \chi \gg 1, \\ \chi^{3/2}, & \text{when } \chi \ll 1. \end{cases} \quad (18)$$

The limit $\chi \gg 1$ recovers Einstein's general relativity and the condition $\chi \ll 1$ yields a relativistic version of MOND. In this last regime, the first two terms on the right-hand side of the trace (15) are smaller than the third and so (cf. Bernal et al. 2011b)

$$f'(\chi)\chi - 2f(\chi) \ll 3L_M^2 \Delta f'(\chi), \quad (19)$$

at all orders of approximation, and so the trace (15) is given by

$$3L_M^2 \Delta f'(\chi) = \frac{8\pi GL_M^2}{c^4} T. \quad (20)$$

Since we are interested in the field produced by a point mass M , then the right-hand side of equations (13) and (20) are null away from the source and so, the last relation in vacuum can be rewritten as

$$\Delta f'(\chi) = 0. \quad (21)$$

As shown by Bernal et al. (2011b), the relation $f(\chi) = \chi^{3/2}$ yields the correct MONDian non-relativistic limit. However, for the sake of generality we will assume in what follows that the function $f(\chi)$ is of power-law form 17. In this case, relation (21) is equivalent to

$$\Delta f'(R) = 0, \quad (22)$$

to all orders of approximation for a power-law function of the Ricci scalar

$$f(R) = R^b. \quad (23)$$

Substitution of the power-law function (17) in the null variations of the gravitational field's action (8) in vacuum means that

$$\delta S_f = -\frac{c^3}{16\pi G} L_M^{2(b-1)} \delta \int R^b \sqrt{-g} d^4x = 0, \quad (24)$$

and so

$$\delta \int R^b \sqrt{-g} d^4x = 0. \quad (25)$$

This equation gives the same field equations as the null variation of the action for a standard power-law metric $f(R)$ theory (23) in vacuum. With this in mind, we can follow the standard perturbation analysis for $f(R)$ restricted by the constraint equation (22) needed to yield the correct MOND-like limit. Since we are only interested in a power-law description of gravity far away from general relativity (cf. equation 18), then in what follows we use the standard $f(R)$ field equations for vacuum as described by Capozziello & Faraoni

(2011) for a power-law description of gravity given by equation (23) with $b = 3/2$, with the constraint (22). To follow their notation, we write the field equations (13) in vacuum as

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = 0, \quad (26)$$

where the fourth-order terms are grouped into the following quantity:

$$\mathcal{H}_{\mu\nu} := -(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta) f'(R). \quad (27)$$

The trace of equation (26) is thus given by

$$f'(R)R - 2f(R) + \mathcal{H} = 0, \quad (28)$$

with

$$\mathcal{H} := \mathcal{H}_{\mu\nu} g^{\mu\nu} = 3\Delta f'(R). \quad (29)$$

The mathematical form of the field's action (8) includes the Schwarzschild mass (through L_M) in the description of the gravitational field. This is usually not the case for the description of the gravitational field since the matter content is generally assumed to appear only in the matter action (12). Following the remarks by Sobouti (2007) and Mendoza & Rosas-Guevara (2007), where similar conclusions were reached, one should expect extensions to the theory even at the fundamental level of the action. For the case of a general matter distribution it is not evident what path to follow. As explained by Carranza et al. (2013) and Mendoza (2012), for systems with a high degree of symmetry (such as the Friedmann–Lemaître–Robertson–Walker – FLRW – universe or a spherically symmetric distribution of matter) the action may be postulated as

$$S_f = -\frac{c^3}{16\pi G} \int \frac{f(\chi)}{L_M^2} \sqrt{-g} d^4x, \quad (30)$$

where the mass-energy is given by (see e.g. Misner, Thorne & Wheeler 1973)

$$M(r) = 4\pi \int_0^r \rho(r) r^2 dr. \quad (31)$$

For the case of the FLRW universe, the upper limit of the previous integral is taken as the Hubble radius (cf. Mendoza 2012; Carranza et al. 2013). The connection between the action 30 and the $f(R, T)$ theory described by Harko et al. (2011) is then evident through the identification

$$F(R, T) := \frac{f(\chi)}{L_M^2}. \quad (32)$$

The field equations then follow through the full formal variation of the action with respect to both R and T (see e.g. Harko et al. 2011; Mendoza 2012).

The general description of the gravitational theory is by no means complete, and further investigation needs to be carried out in this direction. We only mentioned one possible generalization of the simple point mass description by Bernal et al. (2011b) for completeness. In any case, the lensing phenomena we are interested in occur sufficiently far away from the matter distribution producing it, that these can be correctly described as point mass sources.

In what follows, the sign convention used in the definition of the Riemann tensor becomes a relevant point. As discussed in Appendix A, the solutions to the differential field equations of any $f(R)$ theory of gravity greatly depend on the signature chosen for Riemann's tensor. Two different choices of signature bifurcate on the solution space, a property which does not appear in Einstein's general relativity. This is not surprising as it mirrors the analogous

unfolding of the metric and Palatini approaches in $f(R)$ gravity, which does not appear in Einstein's $f(R) = R$ theory (see e.g. Olmo 2011). Throughout the article, we select a particular branch of solutions given by the nowadays almost standard definition of Riemann's tensor in equation (A3).

In dealing with some of the cumbersome algebraic manipulations that a perturbation to an $f(R)$ theory of gravity presents, we have used the Computer Algebra System (CAS) Maxima to facilitate the computations. The Metric EXtended-gravity Incorporated through a Computer Algebraic System (MEXICAS) code (copyright of TB, SM and LAT and licensed with a GNU Public License Version 3) we wrote for this is described in Appendix B and can be downloaded from <http://www.mendoza.org/sergio/mexicas>. Furthermore, development on the treatment of the field equations by the MEXICAS code is described in Appendix C.

For the case of a static spherically symmetric space-time (1) it follows that

$$\mathcal{H}_{\mu\nu} = -f'' \left\{ R_{,\mu\nu} - \Gamma_{\mu\nu}^1 R_{,r} - g_{\mu\nu} \left[\left(g_{,r}^{11} + g^{11} (\ln \sqrt{-g})_{,r} \right) \times R_{,r} + g^{11} R_{,rr} \right] \right\} - f''' \left\{ R_{,\mu} R_{,\nu} - g_{\mu\nu} g^{11} R_{,r}^2 \right\}, \quad (33)$$

and

$$\mathcal{H} = 3f'' \left[\left(g_{,r}^{11} + g^{11} (\ln \sqrt{-g})_{,r} \right) R_{,r} + g^{11} R_{,rr} \right] + 3f''' g^{11} R_{,r}^2. \quad (34)$$

Under the assumption of spherical symmetry, the angular terms of the metric are not perturbed and so

$$\sqrt{-g} = r^2 \sin \theta \left\{ 1 + \left[g_{00}^{(2)} - g_{11}^{(2)} \right] + \mathcal{O}(4) \right\}^{1/2}, \quad (35)$$

then, by using the fact that $\ln(\sqrt{-g})_{,r} = (\sqrt{-g})_{,r} / \sqrt{-g}$, it follows that

$$\ln(\sqrt{-g})_{,r} = \frac{2}{r} + \frac{1}{2} \left[g_{00,r}^{(2)} - g_{11,r}^{(2)} \right] + \mathcal{O}(4). \quad (36)$$

Since Ricci's scalar depends on the metric components and their derivatives up to the second order with respect to the coordinates, it follows it can only have a non-null second and higher perturbation orders, i.e.

$$R = R^{(2)} + R^{(4)} + \mathcal{O}(6). \quad (37)$$

The fact that $R^{(0)} = 0$ is consistent with the flatness of space-time assumption at the lowest zeroth order of perturbation. The expression for the second order component of Ricci's scalar from the metric components (6) is given by

$$R^{(2)} = -\frac{2}{r} \left[g_{11,r}^{(2)} + \frac{g_{11}^{(2)}}{r} \right] - g_{00,rr}^{(2)} - \frac{2}{r} g_{00,r}^{(2)}. \quad (38)$$

The global minus sign that appears on the right-hand side of equation (38) for Ricci's scalar $R^{(2)}$ at second perturbation order differs from that reported by Capozziello, Stabile & Troisi (2007), Capozziello & Stabile (2009). As mentioned above, and discussed in Appendix A, this fact occurs due to the choice of signs in the definition of Riemann's tensor. The particular choice used throughout the article is the one given by equation (A3) and so, our solutions lie in a different branch as the one reported by those authors.

4 LOWEST ORDER SOLUTION

Let us now calculate the order of the trace equation (28) using relations (23) and (37). On the one hand, the lowest order of the first

two terms on the left-hand side of the trace equation is $\mathcal{O}(2b)$. On the other hand, direct inspection of the right-hand side of equation (34) results in the fact that the lowest order of \mathcal{H} is $\mathcal{O}(2b - 2)$. Indeed, the last term of the right-hand side of this equation is $\propto f''' g^{11} R_{,r}^2$ and so, to the lowest order of perturbation of relations (7) and (37), this means that \mathcal{H} contains terms of the form $R^{(2)b-3} R_{,r}^{(2)2}$ and so, \mathcal{H} is of the order of $\mathcal{O}(2b - 2)$. This analysis indicates that to the lowest order the trace equation to consider is

$$\mathcal{H}^{(2b-2)} = 3\Delta f^{(2b-2)}(R) = 0. \quad (39)$$

This result is consistent with relation (22) to the lowest order of approximation and is in perfect agreement with the perturbative study performed by Bernal et al. (2011b). Note also that this is the only independent equation at this order.

Direct substitution of equations (23) and (37) into the last equation leads to

$$\mathcal{H}^{(2b-2)} = 3b(b-1)R^{(2)b-2}g^{11(0)} \left[(\ln \sqrt{-g})_{,r}^{(0)} R_{,r}^{(2)} + R_{,rr}^{(2)} \right] + 3b(b-1)(b-2)R^{(2)b-3}g^{11(0)} R_{,r}^{(2)2} = 0. \quad (40)$$

Substitution of expressions (7) and (36) in the previous equation leads to the following differential equation for Ricci's scalar at order $\mathcal{O}(2)$:

$$R^{(2)} \left[\frac{2}{r} R_{,r}^{(2)} + R_{,rr}^{(2)} \right] + (b-2)R_{,r}^{(2)2} = 0, \quad (41)$$

which can be written in a more suitable form as

$$[\ln R_{,r}^{(2)}]_{,r} + (b-2)[\ln R^{(2)}]_{,r} = -\frac{2}{r}. \quad (42)$$

The solution of the previous equation is

$$R^{(2)}(r) = \left[(b-1) \left(\frac{A}{r} + B \right) \right]^{1/(b-1)}, \quad (43)$$

where A and B are constants of integration.

Far away from the central mass, space-time is flat and so Ricci's scalar must vanish at large distances from the origin. This means that the constant $B = 0$ and so

$$R^{(2)}(r) = \left[(b-1) \frac{A}{r} \right]^{1/(b-1)}. \quad (44)$$

As explained by Bernal et al. (2011b), the case $b = 3/2$ yields a MOND-like weak field limit and so, substituting $b = 3/2$ in relation (44) yields:

$$R^{(2)}(r) = \frac{\hat{R}}{r^2}, \quad (45)$$

where $\hat{R} := A^2/4$. This is exactly the same result as the one obtained by Bernal et al. (2011b). As these authors have shown, this result yields a MONDian-like behaviour for the gravitational field in the limit $r \gg l_M \gg r_g$. For this particular case, the lowest order of approximation of the theory is $\mathcal{O}(1)$, which has a higher relevance as compared to the order $\mathcal{O}(2)$ of standard general relativity for which $b = 1$. Using very general arguments, the authors also showed that the constant $\hat{R} \propto r_g/l_M$ and so, \hat{R} is proportional to the square root of the mass of the central object. In order to calculate \hat{R} from perturbation analysis we need to find the expressions for the metric at order $\mathcal{O}(2)$ of approximation.

5 $f(\chi) = \chi^{3/2}$ METRIC COMPONENTS

Let us now solve the field equations at the next order $\mathcal{O}(2b)$ of approximation. At this order, we expect the metric components $g_{00}^{(2)}$,

$g_{11}^{(2)}$ and Ricci's scalar $R^{(4)}$ to play a relevant role in the description of the gravitational field. In fact, the field equations (26) at this order are given by

$$bR^{(2)b-1}R_{\mu\nu}^{(2)} - \frac{1}{2}R^{(2)b}g_{\mu\nu}^{(0)} + \mathcal{H}_{\mu\nu}^{(2b)} = 0, \quad (46)$$

where

$$\mathcal{H}_{\mu\nu}^{(2b)} = -(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta) f^{(2b)}(R). \quad (47)$$

The complete $\mathcal{H}_{\mu\nu}^{(2b)}$ from equation (33) is written in Appendix C.

Now, from equation (22) it follows that the Laplace–Beltrami operator applied to $f'(R)$ must be zero at all perturbation orders. In particular $\Delta f^{(2b)} = 0$. With this condition, the field equations (46) simplify greatly and can be written as

$$\begin{aligned} bR^{(2)b-1}R_{\mu\nu}^{(2)} - \frac{1}{2}R^{(2)b}g_{\mu\nu}^{(0)} \\ - b(b-1)\{R^{(2)b-2}[R_{\mu\nu}^{(4)} - \Gamma_{\mu\nu}^{(0)}R_{,r}^{(4)} - \Gamma_{\mu\nu}^{(2)}R_{,r}^{(2)}] \\ + (b-2)R^{(2)b-3}R^{(4)}[R_{\mu\nu}^{(2)} - \Gamma_{\mu\nu}^{(0)}R_{,r}^{(2)}]\} \\ - b(b-1)(b-2)[2R^{(2)b-3}R_{,\mu}^{(2)}R_{,\nu}^{(2)} \\ + (b-3)R^{(2)b-4}R^{(4)}R_{,\mu}^{(2)}R_{,\nu}^{(2)}] = 0. \end{aligned} \quad (48)$$

Direct substitution of the following Christoffel symbols

$$\Gamma_{00}^{(0)} = 0, \quad \Gamma_{00}^{(2)} = -\frac{1}{2}g^{11(0)}g_{00,r}^{(2)}, \quad (49)$$

and relations (6) and (7) in the 00 component of equation (48) leads to

$$bR^{(2)b-1}R_{00}^{(2)} - \frac{1}{2}R^{(2)b} + \frac{1}{2}b(b-1)g_{00,r}^{(2)}R^{(2)b-2}R_{,r}^{(2)} = 0. \quad (50)$$

If we now substitute $b = 3/2$, expression (45) and the value of Ricci's tensor at $\mathcal{O}(2)$ of approximation:

$$R_{00}^{(2)} = -\frac{r g_{00,rr}^{(2)} + 2g_{00,r}^{(2)}}{2r}, \quad (51)$$

into equation (50), we obtain the following differential equation for $g_{00}^{(2)}$:

$$r^2 g_{00,rr}^{(2)} + 3r g_{00,r}^{(2)} + \frac{2\hat{R}}{3} = 0, \quad (52)$$

and so

$$g_{00}^{(2)}(r) = -\frac{\hat{R}}{3} \ln\left(\frac{r}{r_*}\right) + \frac{k_1}{r^2}, \quad (53)$$

where k_1 and r_* are constants of integration. By substitution of this result in equation (38) and using equation (45) we get the following differential equation for $g_{11}^{(2)}$:

$$r g_{11,r}^{(2)} + g_{11}^{(2)} + \frac{k_1}{r^2} + \frac{\hat{R}}{3} = 0, \quad (54)$$

with solution:

$$g_{11}^{(2)}(r) = \frac{k_1}{r^2} + \frac{k_2}{r} - \frac{\hat{R}}{3}, \quad (55)$$

where k_2 is a constant of integration.

6 METRIC COEFFICIENTS FROM ASTRONOMICAL OBSERVATIONS

In this section, we derive the constraints well established by astrophysical phenomenology of asymptotically flat galactic rotation

curves satisfying the Tully–Fisher relation, and the cumulative gravitational lensing observations for elliptical and spiral galaxies and galaxy groups, imply for the metric coefficients for static, spherically symmetric space-times for any metric theory of gravity where dark matter is not required.

To begin with, let us take the radial component (3) of the geodesic equations (2) in the weakest limit of the theory. In this limit, the rotation curve for test particles bound to a circular orbit about a mass M with circular velocity $v(r)$ given by equation (4) is

$$\frac{v^2(r)}{c^2 r} = \frac{1}{2}g^{11}g_{00,r}. \quad (56)$$

Except for the inner regions of spiral galaxies, $v(r)$ can be well approximated by a constant which scales with the fourth root of the total baryonic mass M_b of the spiral galaxy in question, as described by the Tully–Fisher empirical relation (see e.g. Milgrom 2008; Famaey & McGaugh 2012)

$$v = (GM_b a_0)^{1/4}. \quad (57)$$

In fact, it is from numerous observations of galactic rotation curves and total baryonic mass estimates, that the constant a_0 has been calibrated (see e.g. Famaey & McGaugh 2012, and references therein).

We now substitute equations (6) and (7) to order $\mathcal{O}(2)$ of approximation and relation (57) in equation (56) to obtain the following differential equation for $g_{00}^{(2)}$:

$$-g_{00,r}^{(2)} = \frac{2}{r} \left(\frac{v}{c}\right)^2 = \frac{2(GM_b a_0)^{1/2}}{c^2 r}, \quad (58)$$

having as solution

$$\begin{aligned} -g_{00}^{(2)}(r) &= 2 \left(\frac{v}{c}\right)^2 \ln\left(\frac{r}{r_*}\right) \\ &= \frac{2(GM_b a_0)^{1/2}}{c^2} \ln\left(\frac{r}{r_*}\right) = \frac{2r_g}{l_M} \ln\left(\frac{r}{r_*}\right), \end{aligned} \quad (59)$$

where r_* is a scale radius which, from the point of view only of the flat rotation curves of galaxies and the Tully–Fisher relation, remains arbitrary. We therefore see that a necessary and sufficient condition in any metric relativistic theory of gravity, where all observational constraints of galactic rotation curves are satisfied without invoking dark matter, is that $g_{00}^{(2)}$ must satisfy the previous empirically derived relation.

Comparing the theoretical metric coefficient $g_{00}^{(2)}$ given by (53) (obtained from perturbation theory for $f(\chi) = \chi^{3/2}$) and the empirical one (59) (obtained from the phenomenology of flat rotation curves and the Tully–Fisher relation), give the following values for the integration constants needed in equation (53):

$$k_1 = 0, \quad \hat{R} = 6r_g/l_M. \quad (60)$$

In this case, the gravitational potential ϕ from equation (5) takes the form:

$$\phi = -v^2 \ln\left(\frac{r}{r_*}\right) = -(GM_b a_0)^{1/2} \ln\left(\frac{r}{r_*}\right), \quad (61)$$

which yields a radial MONDian acceleration:

$$|a| = |\nabla\phi| = \frac{(GM_b a_0)^{1/2}}{r}. \quad (62)$$

Thus, in the $v/c \ll 1$ limit, the $f(\chi) = \chi^{3/2}$ presented is seen to agree with the observed phenomenology of the observed galactic rotation curves in the absence of dark matter, as already shown by Bernal et al. (2011b).

The g_{11} metric coefficient will be obtained from gravitational lensing phenomenology. We begin from the general deviation angle equation written from the point of view of an observer at infinity, the ‘astronomer’ detecting the gravitational lens in question (see e.g. Weinberg 1972; Schneider, Ehlers & Falco 1992; Keeton & Petters 2005):

$$\beta = 2 \int_{r_i}^{\infty} \frac{[-g_{00}(r)g_{11}(r)]^{1/2} dr}{r [(r/r_i)^2 g_{00}(r_i) - g_{00}(r)]^{1/2}} - \pi, \quad (63)$$

where r_i is the closest approach to the central mass M , and it is related to the impact parameter b through the relation $r_i^2 = b^2 g_{00}(r_i)$.

Over the last few years, it has become clear that the complete phenomenology of gravitational lensing, at the level of extensive massive elliptical galaxies (see e.g. Koopmans et al. 2006; Gavazzi et al. 2007; Barnabè et al. 2011), galaxy groups (see e.g. More et al. 2012), clusters of galaxies (see e.g. Limousin et al. 2007; Newman et al. 2009) and more recently spiral galaxies (see e.g. Dutton et al. 2011; Suyu et al. 2012) can be accurately modelled using total matter distributions having isothermal profiles, when treating the problem from the point of view of Einstein’s general relativity. All these observations show that the dark matter haloes needed to explain gravitational lensing under Einstein’s general relativity obey the same Tully–Fisher scaling with total baryonic mass as the ones needed to explain the observed rotation curves of spiral galaxies. This means that for a given total baryonic mass, spiral and elliptical galaxies and groups of galaxies require dark matter haloes having the same physical properties to explain the observations; from kinematics of rotation curves in the former case to gravitational lensing in the latter one (Dutton et al. 2011; Suyu et al. 2012). Under Einstein’s general relativity the majority of these isothermal matter distribution, particularly at large radii, must be composed of a hypothetical dark matter.

For a static spherically symmetric total matter distribution M_T , since assuming the validity of Einstein’s general relativity Schwarzschild’s metric holds, and therefore $g_{00S} = -1/g_{11S}$, we obtain:

$$g_{00S} = 1 - \frac{2r_g}{r} = 1 - \frac{2GM_T(r)}{c^2 r} = 1 - 2 \left(\frac{v}{c} \right)^2. \quad (64)$$

The subscript S identifies the coefficients of the Schwarzschild metric, and $M_T(r) = v^2 r / G$ refers to the hypothetical isothermal total matter distribution (cf. Binney & Tremaine 2008) needed to explain the observed lensing, when assuming general relativity. From this it follows that the dark matter hypothesis provides a self-consistent interpretation of observed phenomenology: the same dark matter haloes, which are required to explain the observed rotation curves, have been solved for by analysing extensive lensing observations.

From equation (64) it follows that for isothermal total matter haloes under Einstein’s general relativity, the metric coefficient g_{00S} does not depend on the radial coordinate. We can see this by using the empirical Tully–Fisher relation (57) between the velocity and the total baryonic mass in the last identity above. Thus, the coefficient (64) can then be taken outside of the integral (63) of the deviation angle, where for the Schwarzschild metric and isothermal total matter haloes we now obtain

$$\beta = \frac{2}{[1 - 2(v/c)^2]^{1/2}} \int_{r_i}^{\infty} \frac{dr}{r [(r/r_i)^2 - 1]^{1/2}} - \pi. \quad (65)$$

The above radial integral yields $\pi/2$ and we obtain the observed bending angle as

$$\beta = \frac{\pi}{[1 - 2(v/c)^2]^{1/2}} - \pi = \frac{\pi}{[1 - 2(GM_b a_0)^{1/2}/c^2]^{1/2}} - \pi. \quad (66)$$

We see that the well-established empirical result of lensing observations yielding isothermal total dark matter haloes under the standard theory is strictly the observation of constant bending angles which do not depend on the impact parameter, scaling with the observed baryonic total masses as indicated above.

Now, since $(v/c)^2$ is of the order of $\mathcal{O}(2)$, we can write equation (66) as

$$\beta = \pi \left(\frac{v}{c} \right)^2 = \pi \frac{(GM_b a_0)^{1/2}}{c^2} = \pi \frac{r_g}{l_M}. \quad (67)$$

The above equation summarizes all empirical results of gravitational lensing at galactic and galaxy group scales: the bending angle does not depend on the impact parameter and scales with the square root of the total baryonic mass. This last equation gives a clear illustration of the link between the dynamics and the space–time curvature effects induced by the presence of an observed baryonic mass.

We can now use the result of equation (67) to constrain the metric coefficient g_{11} for any metric theory of gravity, seeking an accurate description of the observed gravitational lensing phenomena without the introduction of any hypothetical dark matter. To do this, let us return to the general lensing equation (63), and ask that the results obtained under the Schwarzschild metric with isothermal total matter haloes match those under any metric theory of gravity, at all impact parameters and for any total baryonic masses:

$$\begin{aligned} \frac{1}{2} (\beta + \pi) &= \left[1 + \left(\frac{v}{c} \right)^2 \right] \int_{r_i}^{\infty} \frac{dr}{r [(r/r_i)^2 - 1]^{1/2}} \\ &= \int_{r_i}^{\infty} \frac{[-g_{00}(r)g_{11}(r)]^{1/2} dr}{r [(r/r_i)^2 g_{00}(r_i) - g_{00}(r)]^{1/2}}, \end{aligned} \quad (68)$$

at $\mathcal{O}(2)$ of approximation from equations (63) and (65). Let us rearrange integral (68) in such a way that

$$\begin{aligned} \int_{r_i}^{\infty} \left\{ \left[1 + \left(\frac{v}{c} \right)^2 \right] \frac{1}{r [(r/r_i)^2 - 1]^{1/2}} \right. \\ \left. - \frac{[-g_{00}(r)g_{11}(r)]^{1/2}}{r [(r/r_i)^2 g_{00}(r_i) - g_{00}(r)]^{1/2}} \right\} dr = 0. \end{aligned} \quad (69)$$

Since the result must hold for all impact parameters, the integrand of the above equation must be equal to zero and so

$$\left[1 + \left(\frac{v}{c} \right)^2 \right] \frac{1}{(r/r_i)^2 - 1} = \frac{-g_{00}(r)g_{11}(r)}{(r/r_i)^2 g_{00}(r_i) - g_{00}(r)}. \quad (70)$$

Approximating the previous relation to order $\mathcal{O}(2)$, it follows that the metric coefficient g_{11} is given by

$$g_{11}(r) = - \left[1 + 2 \left(\frac{v}{c} \right)^2 \right] \frac{(r/r_i)^2 [g_{00}(r_i)/g_{00}(r)] - 1}{(r/r_i)^2 - 1}. \quad (71)$$

From a mathematical point of view, since the contribution to the integral in the lensing equation (63) is fully dominated by the region $r \approx r_i$, and given the very mild radial dependence of the empirical g_{00} term, we can take $g_{00}(r_i) \approx g_{00}(r)$ in the above equation to yield:

$$\begin{aligned} g_{11}(r) &= -1 - 2 \left(\frac{v}{c} \right)^2 \\ &= -1 - \frac{2(GM_b a_0)^{1/2}}{c^2} = -1 - 2 \frac{r_g}{l_M}. \end{aligned} \quad (72)$$

Thus, any metric theory of gravity where g_{11} matches the above expression in the regime where gravitational lenses are observed

will accurately reproduce all the observed lensing phenomenology, with M_b the total baryonic mass of the object in question (galaxies or group of galaxies), and no hypothetical dark matter assumed to exist. Equations (59) and (72) give empirical mathematical relations for the metric coefficients at perturbation order $\mathcal{O}(2)$ which reproduce all observed rotation velocity and gravitational lensing phenomenology, without the inclusion of any dark matter component.

Notice that the mass dependence of the second term on the right-hand side in expression (72) for the metric coefficient g_{11} is the same as the factor in expression (59) for g_{00} . This last was obtained for a rigorously flat rotation curve in accordance with the Tully–Fisher relation. This shows that the ratio r_g/l_M of the two important characteristic lengths of the extended metric theory of gravity proposed by Bernal et al. (2011b) is the determinant dimensionless measure of deviations from flat space–time at galactic scales, exactly as expected from the dimensional analysis in Hernandez (2012).

The metric coefficient g_{11} in equation (72) can be directly compared to the results for the $f(\chi) = \chi^{3/2}$ metric theory of Bernal et al. (2011b) obtained in equation (55) with the inclusion of the results of equation (60). This means that the choice of the integration constant

$$k_2 = 0, \quad (73)$$

makes these expressions for the metric component g_{11} identical.

Use of the mathematical approximation $A^x \approx 1 + x \ln A$ to write the following expressions for the full empirical metric coefficients gives:

$$g_{00} \approx 1 + (2r_g/l_M) \ln(r_*/r) \approx (r_*/r)^{2r_g/l_M}, \quad (74)$$

$$g_{11} \approx -1 - (2r_g/l_M) \approx e^{2r_g/l_M}. \quad (75)$$

We note that all the approximations used in this section introduce an error several orders of magnitude smaller than the intrinsic observational uncertainties in the empirical relations used. Therefore, all of the expressions given can be considered as strictly equivalent in regards to the accurate modelling of astrophysical rotation curves and gravitational lensing data.

7 DISCUSSION

By constructing the weak field limit of the metric $f(\chi) = \chi^{3/2}$ theory of gravity developed by Bernal et al. (2011b), we have shown that it is possible to explain both the dynamics of massive particles and the deflection of light by observed astronomical systems such as elliptical galaxies, spiral galaxies and groups of galaxies. Recently, the same metric theory of gravity was shown to be coherent also with the expansion dynamics of the observed Universe (Mendoza 2012; Carranza et al. 2013). This is an expected result from a theory of gravity constructed through astronomical observations: it must be coherent at all scales. The regime of Einstein’s general relativity is by no means violated, since the applications developed in this article ($r \gg l_M$) lie far away from the mass and length scales associated with the ones of Einstein’s general relativity ($r \ll l_M$) (see e.g. Mendoza 2012).

The results of this article were constructed using a static spherically symmetric metric with the time and radial components perturbed up to order $\mathcal{O}(2)$ of approximation. This work generalizes the one of Bernal et al. (2011b) in which the radial metric component was assumed up to order $\mathcal{O}(0)$ only and so, information

on the choice of signature of the Riemann tensor was lost (see Appendix A). Such information is very important while working with fourth-order metric theories of gravity.

We mention again the tremendous importance of a correct choice for the signature of the Riemann tensor as described in Appendix A. The choice (A3), and only that choice, used in this article yields results in agreement with astronomical observations. In other words, astronomical observations fix the correct (and unique) choice of signature for Riemann’s tensor. This is an important result, since otherwise solutions from the other branch appear which are not in accordance with astronomical observations.

Table 1 summarizes our main results. It is important to emphasize that the empirical values of the metric components $g_{00}^{(2)}$ and $g_{11}^{(2)}$ do not depend on any gravitational theory and as such, they represent functions that any successful theory of gravity (such as the one used in this article) needs to match. Notice that observationally, independent empirical constraints fixed the $2/r_g/l_M$ factors in g_{00} and g_{11} to be equal; it is encouraging that the formal mathematical perturbation treatment of the theory proposed also yields identical $\hat{R}/3$ factors in the expressions for g_{00} and g_{11} . If this were not the case, even given the compatible functional forms of empirical and theoretical metric coefficients, the $f(\chi) = \chi^{3/2}$ proposal would have been rejected.

An important fact arises from the usage of the $f(\chi)$ metric theory of gravity and not the $f(R)$ formalism. Although closely related to each other for a power-law function (23) and a mass point source, the correct dimensional approach $f(\chi)$ introduces mass and length scales that, as shown by Bernal et al. (2011b), need to be incorporated into the gravitational field action. Although the field equations in vacuum for both $f(\chi)$ and $f(R)$ under a power-law representation yield the same field equations (since the mass M generating the gravitational field is a constant), $f(R)$ gravity is not capable of reproducing the crucial lensing observations as it lacks a crucial constraint equation (22). The gravitational theory $f(\chi) = \chi^{3/2}$ is able to do so since under this approach the correct limit where MONDian-like effects are expected yield the constraint equation (21) or (22). Notice however that both $f(R)$ and $f(\chi)$ with the

Table 1. The table shows the results obtained for the metric components $g_{00}^{(2)}$ and $g_{11}^{(2)}$ for a static spherical symmetric space–time in scales of galaxies and galaxy groups obtained empirically from astronomical observations of these systems and the ones predicted by the metric $f(\chi) = \chi^{3/2}$ theory of gravity of Bernal et al. (2011b). A good metric theory of gravity must be such that it converges to the inferred values presented in the table. The theory $f(\chi) = \chi^{3/2}$ is in perfect agreement with the observed metric components. The dimensionless ratio formed by the quotient of the gravitational radius r_g to the mass-length scale l_M (see equation 11) is the determinant dimensionless quantity of the problem. Since the metric components determine the ‘gravitational potential’ of the system, the length r_* is undetermined. However, since the natural length scale of the system is l_M one can always assume $r_* = l_M$, which also ensures no sign change in the potential in equation (61) over the domain of applicability $r > l_M$.

Metric coefficient	$g_{00}^{(2)}$	$g_{11}^{(2)}$
Observations	$-\frac{2r_g}{l_M} \ln\left(\frac{r}{r_*}\right)$ (Tully–Fisher)	$-\frac{2r_g}{l_M}$ (lensing)
Theory	$-\frac{\hat{R}}{3} \ln\left(\frac{r}{r_*}\right) + \frac{k_1}{r^2}$	$\frac{k_1}{r^2} + \frac{k_2}{r} - \frac{\hat{R}}{3}$
$f(\chi) = \chi^{3/2}$	$\hat{R} = 6r_g/l_M \quad k_1 = 0$	$\hat{R} = 6r_g/l_M \quad k_2 = 0$

appropriate choice of Riemann's tensor (A3) are able to reproduce the flat rotation curves of galaxies and the Tully–Fisher relation.

In an effort to generalize and look for a fundamental basis to an $f(\chi)$ theory of gravity, Carranza et al. (2013) and Mendoza (2012) have shown that these metric theories are equivalent to the $f(R, T)$ construction of Harko et al. (2011). These authors have also shown that the particular theory $f(\chi) = \chi^{3/2}$ is in excellent agreement with cosmological observations of SNIa.

An $f(\chi)$ theory of gravity satisfying the limits of equation (18) implies that gravity is no longer scale invariant. In fact, precise gravity tests have been performed only at strong regimes of Einstein's gravity, where $\chi \gg 1$, and so the involved accelerations of test particles are such that $a \gg a_0$ (see e.g. Will 2006). In exactly the opposite regime, where $\chi \ll 1$, where the involved accelerations of test particles are such that $a \lesssim a_0$, gravity differs from Einstein's general relativity. The traditional approach of assuming Einstein's general relativity to be valid at all scales means that unknown dark entities are needed to explain various astrophysical observations. This article heavily reinforces many others (Hernandez et al. 2010, 2012; Bernal et al. 2011a,b; Mendoza et al. 2011; Hernandez & Jiménez 2012; Mendoza 2012; Carranza et al. 2013) that show how astrophysical and cosmological observations can be accounted for without assuming the existence of dark entities and extending gravity so as to be non-scale invariant.

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APPENDIX A: COMMENTS ABOUT THE SIGN CONVENTION IN RIEMANN'S TENSOR

In the study of the gravitational field equations, the link between the curvature of space–time and the matter content is a key fact. All the information regarding the curvature of space–time is contained in the Riemann curvature tensor $R^\alpha_{\beta\eta\theta}$, which is a function of the first and second derivatives of the metric. From a purely mathematical point of view, the Riemann tensor can be obtained from the Commutator of covariant derivatives (Carroll 2004):

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma, \quad (\text{A1})$$

for any vector field V^α . From a geometrodynamical point of view, the curvature tensor is constructed through the change ΔA_μ in a vector A_μ after being displaced about any infinitesimal closed contour (Landau & Lifshitz 1975): $\Delta A_\mu = \oint \Gamma^\lambda_{\mu\nu} A_\lambda dx^\nu$. By the use of

Stokes' theorem it then follows that for a sufficiently small closed contour:

$$\Delta A_\mu \approx \frac{1}{2} R^\lambda_{\mu\nu\theta} A_\lambda \Delta f^{\nu\theta}, \quad (\text{A2})$$

where $\Delta f^{\nu\theta}$ represents the infinitesimal area enclosed by the contour of the line integral. In this respect, it follows that the Riemann tensor measures the curvature of space-time (cf. Landau & Lifshitz 1975).

In equations (A1) and (A2), the Riemann tensor has been defined as

$$R^\beta_{\mu\nu\alpha} := \Gamma^\beta_{\mu\alpha,\nu} - \Gamma^\beta_{\mu\nu,\alpha} + \Gamma^\beta_{\lambda\nu} \Gamma^\lambda_{\mu\alpha} - \Gamma^\beta_{\lambda\alpha} \Gamma^\lambda_{\mu\nu}. \quad (\text{A3})$$

If Riemann's tensor is defined by equation (A3), then Ricci's tensor is $R_{\nu\alpha} := g^{\beta\mu} R_{\beta\mu\nu\alpha}$ and Ricci's scalar is R^α_α . Since these are the most used definitions in relativity theory nowadays, we will refer to these quantities as 'standard'.

However, there is another way in which Riemann's tensor (and Ricci's tensor) can be defined, usually adopted by mathematicians and by CASs such as Maxima (<http://maxima.sourceforge.net>). In these cases, the syntax is such that (see e.g. Toth 2005)

$$R[\mu, \nu, \alpha, \beta] := R^\beta_{\mu\nu\alpha} = \Gamma^\beta_{\mu\nu,\alpha} - \Gamma^\beta_{\mu\alpha,\nu} + \Gamma^\beta_{\lambda\alpha} \Gamma^\lambda_{\mu\nu} - \Gamma^\beta_{\lambda\nu} \Gamma^\lambda_{\mu\alpha}. \quad (\text{A4})$$

If Riemann's tensor is defined by equation (A4), then Ricci's tensor is $R_{\nu\alpha} := g^{\beta\mu} R_{\beta\mu\nu\alpha}$ and Ricci's scalar is R^α_α . Although this choice of signs for the Riemann and Ricci tensors is not very much in use these days, some well-known textbooks use them (see e.g. the table of sign conventions at the beginning of reference Misner et al. 1973). The CAS Maxima uses the definition (A4) and is such that

$$R_{\text{maxima}} = -R_{\text{standard}}, \quad (\text{A5})$$

in free-index notation.

As discussed in the table of sign conventions of Misner et al. (1973), general relativity can use any of the above definitions (and a few more) simply because of the linearity with which Ricci's scalar and Ricci's tensor appear in Einstein's field equations. This is however not the case in metric $f(R)$ theories of gravity, since for example in those theories, the trace of the field equations is given by (see e.g. Capozziello & Faraoni 2011)

$$f'(R)R - 2f(R) + 3\Delta f(R) = \frac{8\pi G}{c^4} T. \quad (\text{A6})$$

To highlight the point, let us substitute the power-law function (23) in the previous equation to obtain

$$(b-2)R^b + 3b\Delta R^{b-1} = \frac{8\pi G}{c^4} T. \quad (\text{A7})$$

This equation reflects a crucial fact about the choice of sign in Riemann's tensor. Due to the presence of the derivative term $f'(R) = bR^{b-1}$, depending on the sign convention of the definition of the Ricci scalar, there appears a sign factor $(\pm)^{b-1}$ which is not global to all the terms in the equation. This establishes a *bifurcation* in this class of solutions of the theory. Indeed, for a situation where $f(R) = R^a + R^b$ or any more complicated function of R , there is not (a priori) any indication of which convention in the definition of Riemann's tensor should be used to describe a particular physical phenomena. In this article, we show that, under the theory being presented, the convention can be settled. The results presented in this article were obtained with the standard definition of Riemann's tensor in equation (A3). That choice (and only that one) can account for both observed dynamics of massive particles in spiral galaxies through the Tully–Fisher relation,

and for the deflection of light observed in gravitational lenses. An important aspect to point out is that the case $f(R) = R$ of Einstein's general relativity is free from the above ambiguity. This is so because it is possible to redefine the signature for the energy-momentum tensor to recover the same field equations (see e.g. Misner, Thorne & Wheeler 1973; Hobson, Efstathiou & Lasenby 2006).

We see from this result that previous works by Capozziello et al. (2007), Capozziello & Stabile (2009) have selected the convention used by the CAS Maxima in order to compute their results. In that respect, their results lie in another branch of the solutions of the field equations. If we would have taken for example, the definition of Riemann's tensor by Maxima, then the metric coefficients would have been: $g^{(2)}_{00} = 2\hat{R} \ln^2(r)/9 + A \ln(r) + B$ and $g^{(2)}_{11} = -2\hat{R} \ln(r)/9 + D/r + (\hat{R} - A)/2$ (where A , B and D are constants). These are very different from the ones obtained in equations (53) and (55) and would have never reproduced the astrophysical observations treated in this article. It is only through the correct choice of signs in the definition of Riemann's tensor, such as the ones used in the present article and represented in equation (A3), that the good agreement with the Tully–Fisher relation and lensing observations can be correctly obtained.

APPENDIX B: COMMENTS ABOUT THE MAXIMA CODE

In this section, we give a brief introduction to the code we wrote in the CAS Maxima (<http://maxima.sourceforge.net>) to obtain the field equations. Specifically, we work with the module `ctensor` (cf. Toth 2005). The syntax of such module is that, when invoked, it runs an input interface to design the form of the covariant metric.

The Maxima code `MEXICAS` is copyright of TB, SM and LAT, licensed under a GNU Public General License (GPL), version 3 (see <http://www.gnu.org/licenses>) can be obtained from <http://www.mendoza.org/sergio/mexicas> (see the section about copyright and usage in that web page).

For the implementation of the code, we consider a perturbative approach in the parameter $\epsilon := 1/c$, such that the covariant components of the metric are given by

$$\begin{aligned} g_{00} &= 1 + \epsilon^2 g^{(2)}_{00} + \mathcal{O}(4), \\ g_{11} &= -1 + \epsilon^2 g^{(2)}_{11} + \mathcal{O}(4), \end{aligned} \quad (\text{B1})$$

where the angular components are given by the standard expressions for spherical coordinates as shown in equation (6). With these equations, it is simple to construct the contravariant components of the metric:

$$\begin{aligned} g^{00} &= 1 - \epsilon^2 g^{(2)}_{00} + \mathcal{O}(4), \\ g^{11} &= -1 - \epsilon^2 g^{(2)}_{11} + \mathcal{O}(4). \end{aligned} \quad (\text{B2})$$

With these considerations, the metric is recorded in the `ctensor` module. From this fact, it is simple to invoke all the quantities required to construct the field equations, either in general relativity or for any extended metric theory of gravity. For example, in a descriptive way concerning the syntax of maxima it follows that

$$\text{christof}(mcs) \longrightarrow \Gamma^\lambda_{\mu\nu}, \quad (\text{B3})$$

and with similar syntax for the Riemann tensor, the Ricci tensor and the Ricci scalar.

Due to the fact that the metric has an order parameter ϵ , all the tensorial quantities involved in the construction of the field

equations will gain this dependence. In the formalism of the code, it is a crucial fact to extract the perturbation order of every metric quantity to construct the field equations at the desired perturbation order. For example, for a generic quantity q calculated from the manipulation of the metric, if we consider that $q^{(n)}$ represents such quantity at order n , we have

$$q^{(0)} = \lim_{\epsilon \rightarrow 0} q, \quad (\text{B4})$$

which reproduces the flat space-time limit. For the second order we have

$$q^{(2)} = \lim_{\epsilon \rightarrow 0} \frac{q - q^{(0)} - \epsilon q^{(1)}}{\epsilon^2}, \quad (\text{B5})$$

and consequently the fourth order is obtained by

$$q^{(4)} = \lim_{\epsilon \rightarrow 0} \frac{q - q^{(0)} - \epsilon q^{(1)} - \epsilon^2 q^{(2)} - \epsilon^3 q^{(3)}}{\epsilon^4}. \quad (\text{B6})$$

Similarly, higher perturbation orders can be obtained by the obvious generalization of the previous relation.

In equations (B5) and (B6), it is implied that the first-order quantities vanish, as is also the case for the Christoffel symbols. This computational procedure gives as an output a key result used in the article corresponding to Ricci's scalar at second perturbation order, given by equation (38).

APPENDIX C: EXTENDED FIELD EQUATIONS USING MAXIMA

By using the CAS Maxima and the MEXICAS code (see appendix B), we obtained the field equations up to the second order.

The trace (28) of the field equations (46) to the order $\mathcal{O}(2b)$ of approximation can be simplified with the aid of the solutions found at the lowest order of approximation in Section 4 to obtain

$$\begin{aligned} (b-2)R^{(2)4} - 3b(b-1)R^{(2)} \left\{ R^{(2)} \left[R_{,rr}^{(4)} + \frac{2}{r} R_{,r}^{(4)} \right] \right. \\ \left. + \frac{1}{2} R_{,r}^{(2)} \left(g_{00,r}^{(2)} + g_{11,r}^{(2)} \right) \right\} \\ + 3b(b-1)(b-2)R_{,r}^{(2)2} R^{(4)} = 0. \end{aligned} \quad (\text{C1})$$

The components $\mathcal{H}_{\mu\nu}^{(2b)}$ of the field equations (26) at order $\mathcal{O}(2b)$ are given by

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(2b)} = -b(b-1) \left\{ R^{(2)b-2} \left[R_{,\mu\nu}^{(4)} - \Gamma_{\mu\nu}^{(0)} R_{,r}^{(4)} - \Gamma_{\mu\nu}^{(2)} R_{,r}^{(2)} \right. \right. \\ \left. - g_{\mu\nu}^{(0)} \left(R_{,r}^{(2)} \left[g_{,r}^{11(2)} + g^{11(0)} (\ln \sqrt{-g})_{,r}^{(2)} + g^{11(2)} \right. \right. \right. \\ \left. \left. \times (\ln \sqrt{-g})_{,r}^{(0)} \right] + g^{11(0)} \left[(\ln \sqrt{-g})_{,r}^{(0)} R_{,r}^{(4)} + R_{,rr}^{(4)} \right] \right. \\ \left. + g^{11(2)} R_{,rr}^{(2)} \right) - g_{\mu\nu}^{(2)} g^{11(0)} \left(R_{,r}^{(2)} (\ln \sqrt{-g})_{,r}^{(0)} + R_{,rr}^{(2)} \right) \\ \left. + (b-2)R^{(2)b-3} R^{(4)} \left[R_{,\mu\nu}^{(2)} - \Gamma_{\mu\nu}^{(0)} R_{,r}^{(2)} - g_{\mu\nu}^{(0)} g^{11(0)} \right. \right. \\ \left. \left. \times (\ln \sqrt{-g})_{,r}^{(0)} R_{,r}^{(2)} + R_{,rr}^{(2)} \right] \right\} - b(b-1)(b-2) \\ \times \left\{ R^{(2)b-3} \left[2R_{,\mu}^{(2)} R_{,\nu}^{(4)} - g_{\mu\nu}^{(0)} \left(2g^{11(0)} R_{,r}^{(2)} R_{,r}^{(4)} + g^{11(2)} \right. \right. \right. \\ \left. \left. \times R_{,r}^{(2)2} \right) - g_{\mu\nu}^{(2)} g^{11(0)} R_{,r}^{(2)2} \right] + (b-3)R^{(2)b-4} R^{(4)} \\ \left. \times \left[R_{,\mu}^{(2)} R_{,\nu}^{(2)} - g_{\mu\nu}^{(0)} g^{11(0)} R_{,r}^{(2)2} \right] \right\}. \end{aligned} \quad (\text{C2})$$

Dividing the field equations (46) by $R^{(2)b-4}$ and using the trace (C1) and the last equation, a reduced expression for the field equations is found:

$$\begin{aligned} -\frac{2b-1}{6} R^{(2)4} + bR^{(2)3} R_{\mu\nu}^{(2)} - b(b-1)R^{(2)2} \\ \times \left[R_{,\mu\nu}^{(4)} - \Gamma_{\mu\nu}^{(0)} R_{,r}^{(4)} - \Gamma_{\mu\nu}^{(2)} R_{,r}^{(2)} \right] - b(b-1)(b-2)R^{(2)} \\ \times \left[R_{,\mu\nu}^{(4)} \left(R_{,\mu\nu}^{(2)} - \Gamma_{\mu\nu}^{(0)} R_{,r}^{(2)} \right) + 2R_{,\mu}^{(2)} R_{,\nu}^{(4)} \right] \\ - b(b-1)(b-2)(b-3)R_{,\mu}^{(2)} R_{,\nu}^{(2)} R^{(4)} = 0, \end{aligned} \quad (\text{C3})$$

which can also be regarded as the traceless component of the field equations.

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