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Collapse of a radiating star revisited

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Summary. Some further results are obtained concerning the model of a collapsing star studied by de Oliveira *et al.* We study this model without imposing any particular initial static configuration.

1 Introduction

One of us (Santos 1985) studied the junction conditions of a collapsing spherically symmetric non-adiabatic fluid which produces a pure radiation field. The collapsing fluid is isotropic with dissipation under a radial heat flow. The interior spacetime where the fluid is described is assumed shear-free and the exterior spacetime is given by Vaidya's (1953) metric which represents the radial flow of a pure radiation field.

A particular solution of this model has been proposed by de Oliveira, Santos & Kolassis (1985) and de Oliveira *et al.* (1986). It describes an initial static configuration which has been considered to be the Schwarzschild interior solution evolving until the horizon is formed. Bonnor (1987) studied the arrow of time for that solution.

In this paper we study again this same solution but without imposing any particular initial static configuration. It is shown that some of the properties of this model depend only upon its initial mass and the initial radius of the fluid distribution.

2 Vaidya's metric

Vaidya's (1953) metric

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathfrak{r}}\right) dv^2 - 2dv\, d\mathfrak{r} + \mathfrak{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (1)$$

with coordinates $x^0 = \omega$, $x^1 = \mathfrak{r}$, $x^2 = \theta$, $x^3 = \phi$ represents an outgoing pure radiation filled region exterior to a gravitating spherical body. The quantity $m(v)$ is interpreted as the Newtonian mass

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of the body measured by an observer at infinity. If m is constant then by the introduction of the time coordinate $t = \mathbb{r} + v + 2m \ln|\mathbb{r} - 2m|$ (Eddington 1924; Finkelstein 1958) the metric (1) is cast into a form where at once one recognizes the Schwarzschild line element. The energy–momentum tensor associated with equation (1) and describing the pure radiation field is

$$kT_{\alpha\beta} = \frac{2}{\mathbb{r}^2} L \delta^0_\alpha \delta^0_\varrho, \tag{2}$$

where k is the gravitational constant and

$$L = -\frac{dm}{dv} \tag{3}$$

is interpreted as the total luminosity perceived by an observer at infinity (Lindquist, Schwartz & Misner 1965). This suggests that the only physically acceptable situations are those for which $m(v)$ is a non-increasing function of the retarded time v, i.e.

$$\frac{dm}{dv} \leqslant 0. \tag{4}$$

This is also compatible with the fact that the body loses energy due to the outgoing radiation. The magnitude of the vector

$$n_\mu = \delta^1_\mu - 2\frac{dm}{dv}\delta^0_\mu, \tag{5}$$

normal to the hypersurface $\mathbb{r} = 2m(v)$ is

$$n_\mu n^\mu = 4\frac{dm}{dv}. \tag{6}$$

It follows that for physically tenable situations where condition (4) is satisfied this hypersurface is always spacelike while in the limiting case $dm/dv = 0$ it becomes null. Because of this in every situation concerning the collapse or the expansion of a radiating body the hypersurface $\mathbb{r} = 2m(v)$ lies in the interior metric region where Vaidya's metric no longer holds (Lindquist *et al.* 1965).

By assuming that $m(v)$ does not vanish for a finite v and because it is positive and non-increasing with respect to v, at $v = +\infty$ we must have $dm \leqslant dv = 0$. It follows that the hypersurface $\mathbb{r} = 2m(+\infty)$ is null. It is interesting also to note that from equation (3) the luminosity at $v = +\infty$ vanishes, $L(+\infty) = 0$. We can easily explain this by showing that at $v = +\infty$ the radiation emitted by the body suffers an infinite redshift with respect to an observer at rest at infinity (Lindquist *et al.* 1965; de Oliveira *et al.* 1985, 1986). In fact, $\mathbb{r} = 2m(+\infty)$ constitutes the future apparent horizon of the external observer. Due to the existence of the outgoing radiation this hypersurface cannot describe a black hole horizon. To study the history of the body after $v = +\infty$ it is imperative that one finds an analytic extension of the Vaidya metric analogous to the analytic extension found by Kruskal (1960) for the Schwarzschild spacetime. Some progress in this direction has already been made (Israel 1967; Waugh & Lake 1986).

3 Model of a collapsing and radiating star

We consider as in the paper by de Oliveira *et al.* (1985) a spherical body which acts in its exterior as the source of a Vaidya spacetime described by equation (1), and generates in its interior a spherically symmetric spacetime whose line element in isotropic coordinates reads

$$ds^2 = -A^2(r, t)\,dt^2 + B^2(r, t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)], \tag{7}$$

where A and B are assumed to be positive. The energy–momentum tensor of the interior is supposed to be that of a non-viscous shear-free fluid with heat conduction

$$T_{\alpha\beta}=(\mu+p)\,w_\alpha w_\beta+pg_{\alpha\beta}+q_\beta w_\beta+q_\beta w_\alpha, \tag{8}$$

where μ is the energy density of the fluid, p its isotropic pressure, w^α its four-velocity and q^α the radial heat flux vector which has to satisfy, $q_\alpha w^\alpha=0$. The coordinate system (t, r, θ, ϕ) is taken to be comoving, that is

$$w^\alpha=\frac{1}{A}\,\delta^\alpha_0. \tag{9}$$

and thus

$$q^\alpha=q\,\delta^\alpha_1. \tag{10}$$

The Einstein field equations $R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}=kT_{\alpha\beta}$ yield μ, p and q as functions of the metric components

$$k\mu=3\left(\frac{\dot{B}}{AB}\right)^2-\frac{1}{B^2}\left[2\frac{B''}{B}-\left(\frac{B'}{B}\right)^2+\frac{4}{r}\frac{B'}{B}\right], \tag{11}$$

$$kp=\frac{1}{B^2}\left[\left(\frac{B'}{B}\right)^2+\frac{2}{r}\frac{B'}{B}+2\frac{A'}{A}\frac{B'}{B}+\frac{2}{r}\frac{A'}{A}\right]-\frac{1}{A^2}\left[2\frac{\ddot{B}}{B}+\left(\frac{\dot{B}}{B}\right)^2-2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right], \tag{12}$$

$$kq=\frac{2}{AB^2}\left[\left(\frac{\dot{B}}{B}\right)'-2\frac{A'}{A}\frac{\dot{B}}{B}\right], \tag{13}$$

plus an equation with respect to A and B due to the isotropy of the pressure

$$\left(\frac{A'}{A}+\frac{B'}{B}\right)'-\left(\frac{A'}{A}+\frac{B'}{B}\right)^2-\frac{1}{r}\left(\frac{A'}{A}+\frac{B'}{B}\right)+2\left(\frac{A'}{A}\right)^2=0. \tag{14}$$

Here, the dot and the prime stand respectively for differentiation with respect to t and r.

The metrics (1) and (7) can be matched on the spherical hypersurface Σ defined by

$$r=r_\Sigma=\text{const.}, \tag{15}$$

and the junction conditions then yield the following system of equations valid on Σ (Santos 1985)

$$(rB)_\Sigma=\mathfrak{r}_\Sigma, \tag{16}$$

$$p_\Sigma=(qB)_\Sigma, \tag{17}$$

$$[r(rB)']_\Sigma=\left[\frac{\mathfrak{r}}{A}\left(1-\frac{2m}{\mathfrak{r}}\right)\dot{v}+\frac{\mathfrak{r}}{A}\dot{\mathfrak{r}}\right]_\Sigma, \tag{18}$$

$$m(v)=\left[\frac{r^3B}{2A^2}\dot{B}^2-r^2B'-\frac{r^3}{2B}B'^2\right]_\Sigma. \tag{19}$$

From equations (12), (13) and (16)–(19) one can easily obtain the following relations:

$$\dot{v}=\left\{\frac{A^2}{r\dot{B}+A(1+r(\dot{B}/B)]}\right\}_\Sigma, \tag{20}$$

$$\frac{dm}{dv}=-\frac{k}{2}\left(pr^2B^2\frac{A^2}{\dot{v}^2}\right)_\Sigma. \tag{21}$$

The former gives the functional dependence between the retarded time v and the time t, while the last one shows that the mass function $M(v)$ which is determined from the interior metric through equation (19), fulfils condition (4) if and only if the pressure on Σ is non-negative.

Let us now consider solutions of the Einstein field equations for the interior metric of the form

$$A=A_0(r) \tag{22}$$

$$B=B_0(r)\,f(t) \tag{23}$$

where $f(t)$ is positive. Then from equation (14) it follows that A_0 and B_0 are solutions of the equation

$$\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)'-\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)^2-\frac{1}{r}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)+2\left(\frac{A_0'}{A_0}\right)^2=0, \tag{24}$$

describing a static perfect fluid whose energy density μ_0 and pressure p_0 are given by

$$k\mu_0=-\frac{1}{B_0^2}\left[2\left(\frac{B_0'}{B_0}\right)'+\left(\frac{B_0'}{B_0}\right)^2+\frac{4}{r}\frac{B_0'}{B_0}\right], \tag{25}$$

$$kp_0=\frac{1}{B_0^2}\left[\left(\frac{B_0'}{B_0}\right)^2+2\frac{A_0'}{A_0}\frac{B_0'}{B_0}+\frac{2}{r}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)\right]. \tag{26}$$

This static perfect fluid solution matches with the exterior Schwarzschild spacetime and in this case its pressure p_0 vanishes for some value of r. We suppose that this happens for $r=r_\Sigma$

$$(p_0)_\Sigma=0. \tag{27}$$

With the help of equations (22), (23), (25) and (26) equations (11)–(13) can be written

$$k\mu=\frac{1}{f^2}\left(k\mu_0+\frac{3}{A_0^2}\dot{f}^2\right), \tag{28}$$

$$kp=\frac{1}{f^2}\left[kp_0-\frac{1}{A_0^2}(2f\ddot{f}+\dot{f}^2)\right], \tag{29}$$

$$kq=-\frac{2A_0'}{A_0^2B_0^2}\frac{\dot{f}}{f^3}. \tag{30}$$

By substitution of equations (29) and (30) into (17) and by taking equation (27) into account we find

$$2f\ddot{f}+\dot{f}^2-2a\dot{f}=0, \tag{31}$$

where the constant

$$a=\left(\frac{A_0'}{B_0}\right)_\Sigma \tag{32}$$

is positive because the static solution (A_0, B_0) matches with the Schwarzschild exterior metric. The first integral of equation (31) is

$$\dot{f}=-2a\left(\frac{b}{\sqrt{f}}-1\right). \tag{33}$$

By remembering that p_Σ is non-negative and by using equations (27), (29) and (31) we arrive at the important conclusion that the only possible motion of the system is contraction

$$\dot{f}(t) \leq 0. \tag{34}$$

Furthermore, from equation (33) and because $f(t)$ is positive we have

$$0 \leq f(t) \leq b^2. \tag{35}$$

Now, the integration of equation (33) yields

$$t = \frac{1}{a}\left[\tfrac{1}{2}f + b\sqrt{f} + b^2 \ln\left(1 - \frac{\sqrt{f}}{b}\right)\right], \tag{36}$$

where the constant of integration which should enter in equation (36) has been eliminated by means of the transformation $t \rightarrow t + \text{const}$. We observe that the function $f(t)$ decreases monotonically from the value b^2 at $t = -\infty$ to zero at $t = 0$ where a physical singularity is reached.

It follows that the collapse begins at $t = -\infty$ from a static perfect fluid sphere, described by the interior solution $(A_0, b^2 B_0)$, whose energy density and pressure are given by equations (25) and (26) provided that the right-hand members of these equations are multiplied by a factor b^{-2}. From now on, for convenience we absorb the factor b^2 into $B_0(r)$ by setting

$$b = 1. \tag{37}$$

In this way the initial energy density and pressure of the sphere are given by equations (25) and (26), its initial mass by

$$m_0 = -\left(r^2 B_0' + \frac{r^3 B_0'^2}{2B_0}\right)_\Sigma, \tag{38}$$

and its initial radius by

$$\mathfrak{r}_0 = (rB_0)_\Sigma. \tag{39}$$

For $t = -\infty$ the exterior spacetime becomes the Schwarzschild spacetime and by considering the junction conditions we have $A_{0\Sigma}' = m_0/r_{0\Sigma}^2(1 + m_0/2r_{0\Sigma})^2$ and $B_{0\Sigma} = (1 + m_0/2r_\Sigma)^2$, which allows us to express equation (32) in terms of the initial quantities m_0 and $\mathfrak{r}_0$ by using equation (39),

$$a = \frac{m_0}{\mathfrak{r}_0^2}. \tag{40}$$

To obtain the explicit dependence between the retarded time v and the time t we must integrate equation (20). For this we write it in the form

$$\frac{dv}{df} = \left(\frac{A_0^2}{rB_0}\right)_\Sigma \frac{1}{\dot{f}(\dot{f} + h)}, \tag{41}$$

where

$$h = \left[\frac{A_0}{rB_0^2}(rB_0)'\right]_\Sigma. \tag{42}$$

This constant with the help of equations (26), (27), (32) and (39), and (40) can also be written in the form

$$h = \frac{1}{\mathfrak{r}_0}\left(1 - \frac{2m_0}{\mathfrak{r}_0}\right), \tag{43}$$

which shows that h is positive, otherwise the radius of the static sphere would be less than the

Schwarzschild radius. Now, the integral of equation (41) is

$$v=t(\sqrt{f})-\left(\frac{2m_0}{\mathfrak{r}_0}\right)^3 t\left(\frac{\mathfrak{r}_0}{2m_0}\sqrt{f}\right), \tag{44}$$

where $t(\ )$ represents the functional relation defined by (36). From equation (41) and by using (33) we can see that the function $v(f)$ is monotonically decreasing with respect to f, and therefore monotonically increasing with respect to t, in the range $(2m_0/\mathfrak{r}_0)^2<f(t)<1$. From equation (44) we see that $v=-\infty$ at $f=1$ (or $t=-\infty$) and $v=+\infty$ at $f(t_H)=(2m_0/\mathfrak{r}_0)^2$ where the time t_H is given according to equation (36) by

$$t_H=2\mathfrak{r}_0\left(1+\frac{m_0}{\mathfrak{r}_0}\right)+\frac{\mathfrak{r}_0^2}{m_0}\ln\left(1-\frac{2m_0}{\mathfrak{r}_0}\right). \tag{45}$$

The relation between t, f, $\dot{f}$ and v is summarized qualitatively in Table 1.

Table 1.

t	$-\infty$	↗	t_H	↗	0
$\dot{f}$	0	↘	$-\frac{1}{\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)$	↘	$-\infty$
$\sqrt{f}$	1	↘	$\frac{2m_0}{\mathfrak{r}_0}$	↘	0
v	$-\infty$	↗	$+\infty$		

The luminosity L can be calculated from equations (3), (21), (29), (31) and (39)–(41)

$$L=-m_0\left[\mathfrak{r}_0\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)^{-1}\dot{f}+1\right]^2\dot{f}, \tag{46}$$

and the mass m from equations (19), (33), (38), (40), (42) and (43)

$$m=m_0\left[\left(\frac{\mathfrak{r}_0}{2m_0}-1\right)^{-1}\left(1-\sqrt{f}\right)^2+f\right]. \tag{47}$$

By using equations (43) and (47) we can see that at $t=t_H$, or equivalently at $v=+\infty$, the hypersurface Σ satisfies the condition $\mathfrak{r}_H=2m_H$ where $m_H=m(t_H)=2m_0^2/\mathfrak{r}_0$ and therefore it is null. From equation (46) we notice also that at $t=t_H$ we have $L=0$. We conclude that the hypersurface

$$\mathfrak{r}_H=2m_H=\frac{4m_0^2}{\mathfrak{r}_0} \tag{48}$$

constitutes the future event horizon of a distant observer. Clearly, the Vaidya metric (1) cannot describe the exterior spacetime manifold for $\mathfrak{r}<2m$ and for this an analytical extension of this metric is needed.

It is interesting to point out that for the family of solutions (22) and (23) the time for the formation of the horizon (45), as well as its mass and radius (48) do not depend upon the structure of the initial static perfect fluid configuration but only upon its initial radius $\mathfrak{r}_0$ and initial mass m_0.

The differentiation of the luminosity L given by equation (46) with respect to $\dot{f}$ becomes

$$\frac{dL}{d\dot{f}}=-m_0\left[\mathfrak{r}_0\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)^{-1}\dot{f}+1\right]\left[3\mathfrak{r}_0\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)\dot{f}+1\right]. \tag{49}$$

Hence from equations (46) and (49) we can conclude that the luminosity starts increasing from $L_0=0$ at the initial time $\dot{f}=0$ and attains its maximum at

$$\dot{f}=-\frac{1}{3\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right),$$

$$L_{\max}=\frac{4}{27}\frac{m_0}{\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right). \tag{50}$$

After its maximum (50), the luminosity decreases to $L_{\mathrm{H}}=0$ at horizon formation at

$$\dot{f}=-\frac{1}{\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right).$$

We can summarize the above results in Table 2.

Table 2.

$\dot{f}$	0	$\searrow$	$-\frac{1}{3\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)$	$\searrow$	$-\frac{1}{\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)$	$\searrow$	$-\infty$
$\frac{dL}{d\dot{f}}$	$-m_0$	$-$	0	$+$	0	$-$	
L	0	$\nearrow$	$\frac{4m_0}{27\mathfrak{r}_0}\left(1-\frac{2m_0}{\mathfrak{r}_0}\right)$	$\searrow$	0	$\nearrow$	$+\infty$

The energy density μ and the isotropic pressure p have to increase towards the centre of the fluid distribution in order to be physically reasonable. Differentiating equations (28) and (29) with respect to r and utilizing (31), (33) and (37), we obtain

$$k\mu'=k\mu'_0\frac{1}{f^2}-24a^2\frac{A'_0}{A_0^3}\frac{(1-\sqrt{f})^2}{f^3}, \tag{51}$$

$$kp'=kp'_0\frac{1}{f^2}-8a^2\frac{A'_0}{A_0^3}\frac{(1-\sqrt{f})}{f^{5/2}}. \tag{52}$$

From equations (30), (33) and (37) we have

$$kq=\frac{4aA'_0}{A_0^2B_0^2}\frac{(1-\sqrt{f})}{f^{7/2}}, \tag{53}$$

and since we need $q>0$ then we must have $A'_0>0$. Hence from equations (51) and (52) if $\mu'_0<0$ and $p'_0<0$ are satisfied for the static distribution, then the fluid evolves satisfying the physical conditions $\mu'<0$ and $p'<0$.

It has been proved recently (Kolassis, Santos & Tsoubelis 1987) that the energy conditions are fulfilled if the conditions $\mu-3p\geqslant 0$ together with $\mu'<0$ and $p'<0$ are satisfied. From equations

(28), (29), (31), (33) and (37) we can write

$$k\mu - 3kp = \left(k\mu_0 - 3kp_0 + \frac{12a^2}{A_0^2} F \right) \frac{1}{f^2}, \tag{54}$$

where

$$F(f) = \frac{(1-\sqrt{f})(1-2\sqrt{f})}{f}. \tag{55}$$

The function $F(f)$ starts at $F=0$ when $\sqrt{f}=1$ and decreases to a minimum $F=-\frac{1}{4}$ at $\sqrt{f}=\frac{2}{3}$. The metric A_0 decreases to the origin of the distribution, $A_0'>0$. Hence we can be assured that $\mu - 3p \geqslant 0$ is satisfied by the fluid throughout collapse if the initial static configuration satisfies

$$k\mu_0 - 3kp_0 \geqslant \frac{3a^2}{A_0^2(r=0)}. \tag{56}$$

Lastly we observe that the total mass loss of the system for $1 > f(t) > (2m_0/\mathfrak{r}_0)^2$ is

$$\Delta m_- = m_0 - m_{\mathrm{H}} = m_0 \left(1 - \frac{2m_0}{\mathfrak{r}_0} \right), \tag{57}$$

which again we point out, depends only upon its initial configuration. After $f_{\mathrm{H}} = (2m_0/\mathfrak{r}_0)^2$ the total mass of the system starts to increase and we obtain that

$$\Delta m_+ = m(f=0) - m_{\mathrm{H}} = m_0 \left(\frac{2m_0}{\mathfrak{r}_0} \right)^2 \left(1 - \frac{2m_0}{\mathfrak{r}_0} \right)^{-1}. \tag{58}$$

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